Singularities of attached shock waves in steady axially symmetric flow

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An I.B.M. 704 electronic computer was used to integrate the differential equations, which determine the shock singularity, when the velocity after the shock on a body is subsonic. The results are given in twenty-two cases corresponding to three different bodies; the procedure followed was outlined in a previous paper (Cabannes 1952).

1. Introduction

We consider a body of revolution placed in a compressible fluid; the viscosity and thermal conductivity are neglected. The fluid is moving at infinity with a uniform supersonic velocity $\bar{q}$ parallel to the axis of revolution Ox. A shock wave is formed in front of the obstacle; it limits the region in which the flow is uniform. We assume that the nose of the body is a cone of revolution with semi-angle at the apex θ_s, and that the angle θ_s has been chosen such that a shock wave could be attached at the point 0 of the body. For small values of Mach number, $M < M_0(\theta_s)$, the shock wave is detached; for large values of Mach number, $M \geqslant M^{**}(\theta_s)$, the flow after the shock is supersonic; for intermediary values, $M_0(\theta_s) \leqslant M < M^{**}(\theta_s)$, the flow after the shock is subsonic, partially or totally.

When the body is an infinite cone of revolution, the flow after the shock is conical and the shock wave is itself an infinite cone of revolution. Since we assume that only the nose of the body is a cone, the body meridian possesses a rectilinear segment OI, up to the point I, at which point the slope of the tangent, or one of the higher derivatives, is discontinuous; the presence of the discontinuity perturbs the conical flow. Two cases are to be distinguished.

(1) When the speed on the cone is subsonic, that is, for $M_0(\theta_s) \leqslant M < M^{**}(\theta_s)$, the perturbation is propagated in all the region of subsonic flow and then in all the supersonic region if the latter exists. The conical flow is not produced in any region of space and the meridian of the shock wave does not possess a rectilinear segment.

(2) When the speed on the cone is supersonic, that is, for $M^{**}(\theta_s) < M$, the perturbation is propagated along a characteristic IA which meets the shock wave meridian at a point A; this meridian possesses a rectilinear segment OA and a region of space in which conical flow exists.

In the work which follows, we consider only the first case. We locate the position of a point P in a semi-meridian plane by the polar co-ordinates $OP = r$ and $xOP = \theta$ (see figure 1). By means of these co-ordinates, the equation of the semi-cone in which the body terminates is written in the form (1), and the equation of the shock wave in the neighbourhood of the point O is written in the form (2). Thus

$$\text{(body)} \quad \theta = \theta_s, \tag{1}$$

$$\text{(shock)} \quad \theta = \theta_w + Ar^m + \dots. \tag{2}$$

The angle θ_w is determined by the theory of axially symmetric flow (Kopal 1947); it depends on the Mach number M and the angle θ_s. The purpose of the following tables is to determine the value of the exponent m; this exponent depends similarly on the Mach number M and the angle θ_s.

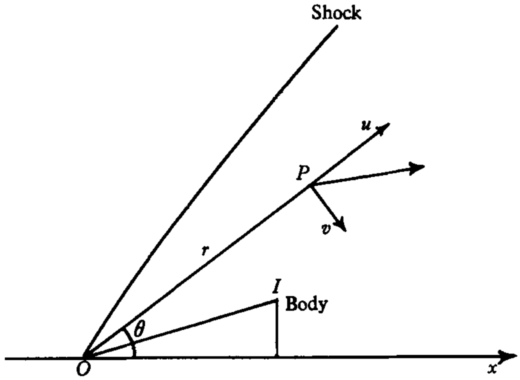

FIGURE 1. Definition of polar co-ordinates.

2. Equations of motion

We designate by u and v the components, in the directions θ and $\theta + \frac{1}{2}\pi$, of the fluid velocity at a point P, by p and ρ the pressure and density of the fluid at this point, and by γ the ratio of the specific heats of the fluid. The four functions u, v, p and ρ of the two variables r and θ satisfy the following partial differential equations which express the fundamental law of dynamics, the conservation of mass and the conservation of energy:

$$\left.\begin{aligned} u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} &= 0, \\ u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} + \frac{1}{r\rho}\frac{\partial p}{\partial \theta} &= 0, \\ \frac{\partial}{\partial r}(r^2\rho u \sin\theta) + \frac{\partial}{\partial \theta}(r\rho v \sin\theta) &= 0, \\ u\frac{\partial}{\partial r}(p\rho^{-\gamma}) + \frac{v}{r}\frac{\partial}{\partial \theta}(p\rho^{-\gamma}) &= 0. \end{aligned}\right\} \tag{3}$$

We attempt to satisfy the preceding equations using functions expanded in series of increasing powers of r; the coefficients depend only on θ. Thus

$$\left.\begin{aligned} u(r,\theta) &= u_0(\theta)+2Ar^m u_m(\theta)+\dots,\\ v(r,\theta) &= v_0(\theta)+2Ar^m v_m(\theta)+\dots,\\ p(r,\theta) &= p_0(\theta)+2Ar^m p_m(\theta)+\dots,\\ \rho(r,\theta) &= \rho_0(\theta)+2Ar^m \rho_m(\theta)+\dots.\end{aligned}\right\} \tag{4}$$

By substitution of the preceding expansions into equations (3) and by identification according to successive powers of r, one obtains an infinite set of differential equations. The equations (3) permit a first integral deduced from the theorem of Bernoulli. Since the limiting speed q_m is constant in front of the shock and continuous across the shock, one can take the following equation to be valid throughout the fluid

$$\frac{2\gamma}{\gamma-1}\frac{p}{\rho}+u^2+v^2 = q_m^2. \tag{5}$$

We now introduce the positive function $a_0(\theta)$ defined by

$$a_0^2 = \frac{\gamma p_0}{\rho_0} = \tfrac{1}{2}(\gamma-1)\,(q_m^2-u_0^2-v_0^2). \tag{6}$$

The differential equations deduced from equations (3) can be written in the following form. Using given initial conditions, the functions with index 0 can be calculated from equations (7) below, whereas the functions with index m can then be calculated, when m has been chosen, from equations (8); the prime indicates differentiation with respect to θ;

$$\left.\begin{aligned} u_0'-v_0 &= 0,\\ v_0'\left(1-\frac{a_0^2}{v_0^2}\right)+u_0\left(1-2\frac{a_0^2}{v_0^2}\right)-\frac{a_0^2}{v_0}\cot\theta &= 0,\\ \frac{\rho_0'}{\rho_0}\left(1-\frac{a_0^2}{v_0^2}\right)+\frac{u_0}{v_0}+\cot\theta &= 0,\\ \frac{p_0'}{p_0} = \gamma\frac{\rho_0'}{\rho_0} &= 0;\end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} u_m' v_0+m u_m u_0 - v_m v_0 + m\frac{a_0^2}{\gamma}\frac{p_m}{p_0} &= 0,\\ v_m'+v_0\left(\frac{\rho_m}{\rho_0}\right)'+(m+2)\,u_m+\left(\cot\theta+\frac{\rho_0'}{\rho_0}\right)v_m+m u_0\frac{\rho_m}{\rho_0} &= 0,\\ m u_0\left(\frac{p_m}{p_0}-\gamma\frac{\rho_m}{\rho_0}\right)+v_0\left(\frac{p_m}{p_0}-\gamma\frac{\rho_m}{\rho_0}\right)' &= 0,\\ \frac{p_m}{p_0}-\frac{\rho_m}{\rho_0}+(\gamma-1)\,\frac{u_0u_m+v_0v_m}{a_0^2} &= 0.\end{aligned}\right\} \tag{8}$$

3. Conditions on the body

The body is formed by the stream surface extending from the point 0. By requiring that the differential equation of the stream surfaces

$$\frac{dr}{u} = \frac{r\,d\theta}{v} \tag{9}$$

be satisfied by the function (1), one obtains the conditions

$$v_0(\theta_s) = 0, \tag{10a}$$

$$v_m(\theta_s) = 0. \tag{10b}$$

4. Conditions on the shock wave

A certain number of conditions must be satisfied across the shock wave. These conditions which express the law of conservation of mass and conservation of energy are stated by the following equations, in which $\bar{c}$, $\bar{p}$ and $\bar{\rho}$ denote the speed of sound, the pressure and density before shock, β the angle which the tangent to the shock wave meridian makes with the axis of revolution, and $\mathscr{M}$ denotes the normal Mach number $M \sin\beta$:

$$\left.\begin{aligned} u &= \bar{q}\cos\theta + \frac{2\bar{c}}{\gamma+1}\left(\mathscr{M} - \frac{1}{\mathscr{M}}\right)\sin(\beta-\theta), \\ v &= -\bar{q}\sin\theta - \frac{2\bar{c}}{\gamma+1}\left(\mathscr{M} - \frac{1}{\mathscr{M}}\right)\cos(\beta-\theta), \\ \frac{p}{\bar{p}} &= \frac{2\gamma}{\gamma+1}\mathscr{M}^2 - \frac{\gamma-1}{\gamma+1}, \\ \frac{\bar{\rho}}{\rho} &= \frac{2}{\gamma+1}\frac{1}{\mathscr{M}^2} + \frac{\gamma-1}{\gamma+1}. \end{aligned}\right\} \tag{11}$$

The Mach number M is expressed as a function of the speed $\bar{q}$ by

$$M^2 = \frac{2}{\gamma-1}\frac{\bar{q}^2}{q_m^2 - \bar{q}^2}. \tag{12}$$

By considering that the relations (11) are satisfied identically on the shock wave, one obtains, for $\theta = \theta_w$,

$$\left.\begin{aligned} u_0(\theta_w) &= \bar{q}\cos\theta_w, \\ v_0(\theta_w) &= \frac{\gamma-1}{\gamma+1}\frac{\bar{q}^2\cos^2\theta_w - q_m^2}{\bar{q}\sin\theta_w}, \\ \frac{p_0(\theta_w)}{\bar{p}} &= \frac{2\gamma}{\gamma+1}M^2\sin^2\theta_w - \frac{\gamma-1}{\gamma+1}, \\ \frac{\bar{\rho}}{\rho_0(\theta_w)} &= \frac{2}{\gamma+1}\frac{1}{M^2\sin^2\theta_w} + \frac{\gamma-1}{\gamma+1}, \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} 2u_m + (m+1)\,u_0\tan\theta_w + (m+1)\,v_0 &= 0, \\ 2v_m + \frac{m(\gamma-3)-4}{\gamma+1}u_0 + \frac{m(\gamma+1)+2}{\gamma+1}v_0\cot\theta_w &= 0, \\ \frac{p_m}{p_0} &= \frac{\gamma}{\gamma+1}\cot\theta_w - (m+1)\frac{2\gamma}{\gamma+1}\frac{u_0 v_0}{a_0^2}, \\ \frac{\rho_m}{\rho_0} &= \frac{m(\gamma+1)+\gamma+2}{\gamma+1}\cot\theta_w + (m+1)\frac{\gamma-1}{\gamma+1}\frac{u_0}{v_0}. \end{aligned}\right\} \tag{14}$$

5. Method of integration

The system (8) has two first integrals, which can be written in the following form, where α denotes a constant, determined by the conditions on the shock wave,

$$\left.\begin{aligned}\frac{p_m}{p_0} &= -\gamma\frac{u_0u_m+v_0v_m}{a_0^2}-\alpha\exp\left\{\int_\theta^{\theta_w}\frac{mu_0}{v_0}d\theta\right\},\\ \frac{\rho_m}{\rho_0} &= -\frac{u_0u_m+v_0v_m}{a_0^2}-\alpha\exp\left\{\int_\theta^{\theta_w}\frac{mu_0}{v_0}d\theta\right\},\\ \text{with}\quad \frac{\alpha}{\gamma(m+1)} &= \frac{u_0^2(\theta_w)}{q_m^2-u_0^2(\theta_w)}\frac{\tan\theta_w}{\gamma-1}-\frac{\gamma+3}{\gamma+1}\frac{\cot\theta_w}{\gamma+1}.\end{aligned}\right\} \tag{15}$$

Thus the system (8) is reduced to a system of two linear non-homogeneous differential equations, that is,

$$\left.\begin{aligned}u_m'-(m+1)\,v_m &= \frac{m}{\gamma}\frac{a_0^2}{v_0}\alpha\exp\left\{\int_\theta^{\theta_w}\frac{mu_0}{v_0}d\theta\right\},\\ \left(1-\frac{v_0^2}{a_0^2}\right)v_m'+Au_m+Bv_m &= \frac{m}{\gamma}u_0\alpha\exp\left\{\int_\theta^{\theta_w}\frac{mu_0}{v_0}d\theta\right\},\end{aligned}\right\} \tag{16}$$

where

$$\left.\begin{aligned}A &= m+2-\frac{mu_0^2+v_0^2}{a_0^2}+(\gamma-1)\frac{u_0v_0^2}{a_0^2}\frac{u_0+v_0\cot\theta}{a_0^2-v_0^2},\\ B &= \frac{2u_0v_0+(a_0^2+v_0^2)\cot\theta}{a_0^2-v_0^2}+(\gamma-1)\frac{v_0^3}{a_0^2}\frac{u_0+v_0\cot\theta}{a_0^2-v_0^2}-2m\frac{u_0v_0}{a_0^2}.\end{aligned}\right\} \tag{17}$$

The general solution is the sum of a particular solution of the complete system plus the general solution of the homogeneous system. For the homogeneous system, all the points of the interval $\theta_s \leqslant \theta \leqslant \theta_w$ are ordinary points; the right-hand sides of (16) are infinite at the extremity $\theta = \theta_s$ of the interval.

In the neighbourhood of the body, we can write the following series expansions

$$\begin{aligned}u_0(\theta) &= u_0(\theta_s)\left\{1-(\theta-\theta_s)^2+\cot\theta_s\frac{(\theta-\theta_s)^3}{3}+\ldots\right\},\\ v_0(\theta) &= u_0(\theta_s)\{-2(\theta-\theta_s)+\cot\theta_s(\theta-\theta_s)^2+\ldots\},\\ \exp\left\{\int_\theta^{\theta_w}\frac{mu_0}{v_0}d\theta\right\} &= O(\theta-\theta_s)^{\frac{1}{2}m},\\ u_m(\theta) &= O(\theta-\theta_s)^{\frac{1}{2}m},\\ v_m(\theta) &= O(\theta-\theta_s)^{\frac{1}{2}m}.\end{aligned}$$

For positive values of the exponent m, the functions u_m, v_m, p_m and ρ_m therefore have finite values on the body. Thus the presence of the singularity in the system (8) for $\theta = \theta_s$ does not introduce any complication in the numerical integration. This singularity was studied by Shen & Lin (1951) for the case $m = 1$. Because they neglected the existence of one of the first integrals, they were led to an indicial equation of third degree which had a double root, and therefore they found a logarithmic term. In reality, the indicial equation is of second degree with two distinct roots. Lin has shown in a private communication that the coefficient of the logarithmic term is zero, which brings the results of Shen & Lin (1951) into accord with those of Cabannes (1951*b*).

To begin the calculations for fixed γ, the angle θ_s of the body is assumed and the two first equations of (7) are integrated starting from $\theta = \theta_s$, taking $v_0(\theta_s) = 0$ and selecting an arbitrary value of $u_0(\theta_s)$. The shock wave is reached when

$$\frac{\gamma-1}{\gamma+1}(q_m^2 - u_0^2) + u_0 v_0 \tan\theta = 0; \tag{18}$$

at this point we have $\theta = \theta_w$, and the Mach number M before the shock is given by the relation

$$M^2 = \frac{2}{\gamma-1}\frac{u_0^2(\theta_w)}{q_m^2\cos^2\theta_w - u_0^2(\theta_w)}. \tag{19}$$

The functions p_0 and ρ_0 are computed starting from the shock conditions by means of the relation (6) and the first integral $p_0\rho_0^{-\gamma} = \text{const}$. The functions of index zero being known, the functions of index m are calculated, for an assumed

θ_s	$\dfrac{u_0(\theta_s)}{q_m}$	θ_w	M	m
10°	0·37	80·056	1·05398	−0·25869
	0·38	75·790	1·05639	0·84264
	0·39	71·640	1·07239	4·18764
	0·40	67·958	1·09481	18·293
20°	0·33	72·869	1·21221	−0·16049
	0·34	70·477	1·21171	0·15595
	0·35	68·096	1·21866	0·61447
	0·36	65·787	1·23144	1·30671
	0·37	63·584	1·24873	2·42736
	0·38	61·500	1·26954	4·45732
	0·39	59·539	1·29315	8·93831
	0·40	57·697	1·31906	25·10423
30°	0·31	69·419	1·48062	−0·04791
	0·32	67·763	1·48228	0·18019
	0·33	66·146	1·48959	0·47582
	0·34	64·577	1·50171	0·86948
	0·35	63·068	1·51792	1·41366
	0·36	61·622	1·53768	2·20664
	0·37	60·241	1·56053	3·45348
	0·38	58·924	1·58613	5·65570
	0·39	57·670	1·61418	10·41496
	0·40	56·477	1·64449	27·46310

TABLE 1

value of m, by integrating the system (8), starting from the relations (14) which are valid for $\theta = \theta_w$. The computation is completed when the value θ_s is reached; the value $v_m(\theta_s)$ thus obtained is in general different from zero. The computation is repeated with the same values of θ_s, $u_0(\theta_s)$ and γ, and a different value of m; successive approximations are used to obtain the value of m for which $v_m(\theta_s) = 0$. We thus define the exponent m as a function of the angle θ_s, the Mach number M, and the adiabatic constant γ. The computations have been carried out with the help of the I.B.M. 704 electronic computer. The numerical integrations were made by the fourth-order Runge–Kutta method; the step length for the integration was chosen to be equal to one-twentieth of a degree.

6. Numerical results

In the computations, the adiabatic constant was taken to be $\gamma = 1{\cdot}4$. Three cones have been considered with $\theta_s = 10°$, $20°$ and $30°$. The results are given in table 1, the angles θ_w being expressed in degrees, and the results are also plotted in figure 2.

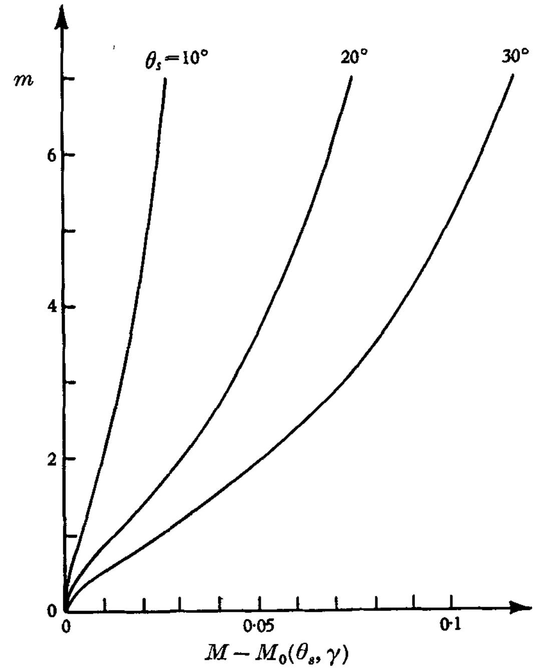

FIGURE 2. Variations of the exponent m for axially symmetric flow.

7. Comparison with the results for plane flow

In the case of plane flow it is possible to calculate the exponent m by means of formulae in finite terms (Cabannes 1951*a*). In the cross-sectional plane of the wedge the axis of symmetry of the wedge is taken as the polar axis and θ_w is the angle between the shock and the polar axis. The equation of the shock in the neighbourhood of the vertex of the wedge can be written, in polar co-ordinates, in the form

$$\theta = \theta_w + Ar^m + \dots. \tag{20}$$

The exponent m is implicitly defined as a function of the wedge angle θ_s, the Mach number M and the adiabatic constant γ by the formulae

$$\left.\begin{aligned}
\Delta \tan m\phi &= \frac{\gamma+1}{4\sin(\theta_w-\theta_s)\cos(\theta_w-\theta_s)} - \frac{\gamma-1}{2}\cot(\theta_w-\theta_s) - \frac{\gamma+1}{4}(\tan\theta_w - \cot\theta_w),\\
\tan\phi &= \Delta\tan(\theta_w-\theta_s),\\
\frac{1}{1-\Delta^2} &= \sin(\theta_w-\theta_s)\cos(\theta_w-\theta_s)\left\{\frac{\gamma+1}{2}\tan\theta_w - \frac{\gamma-1}{2}\tan(\theta_w-\theta_s)\right\},\\
M^2\sin^2\theta_w &= \frac{\tan\theta_w}{\frac{1}{2}(\gamma+1)\tan(\theta_w-\theta_s) - \frac{1}{2}(\gamma-1)\tan\theta_w}.
\end{aligned}\right\} \tag{21}$$

Thirty-five cases have been computed with $\gamma = 1{\cdot}4$. The results are presented in table 2.

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θ_s	θ_w	M	m
10°	67° 26′	1·4210	0·0000
	67°	1·4211	0·1349
	66°	1·4226	0·5536
	65°	1·4255	1·2697
	65°	1·4297	2·8570
	63°	1·4354	14·7041
	62° 52′	1·4363	∞
20°	64° 54′	1·8400	0·0000
	64° 30′	1·8402	0·1915
	64°	1·8411	0·5080
	63° 30′	1·8426	0·9472
	63°	1·8447	1·5999
	62° 30′	1·8474	2·6812
	62°	1·8507	4·8916
	61° 30′	1·8549	13·6217
	61° 19′	1·8591	∞
30°	64° 48′	2·5192	0·0000
	64° 40′	2·5193	0·1357
	64° 30′	2·5195	0·3362
	64° 20′	2·5198	0·5757
	64° 10′	2·5203	0·8661
	64°	2·5210	1·2255
	63° 50′	2·5218	1·6819
	63° 40′	2·5228	2·2792
	63° 30′	2·5239	3·0979
	63° 20′	2·5252	4·2893
	63° 10′	2·5266	5·3123
	63°	2·5282	9·8407
	62° 50′	2·5299	20·6164
	62° 43′	2·5312	∞
40°	66°	4·4473	0·0000
	65° 55′	4·4481	0·5369
	65° 50′	4·4492	1·0597
	65° 45′	4·4504	2·2032
	65° 41′	4·4514	∞

TABLE 2

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